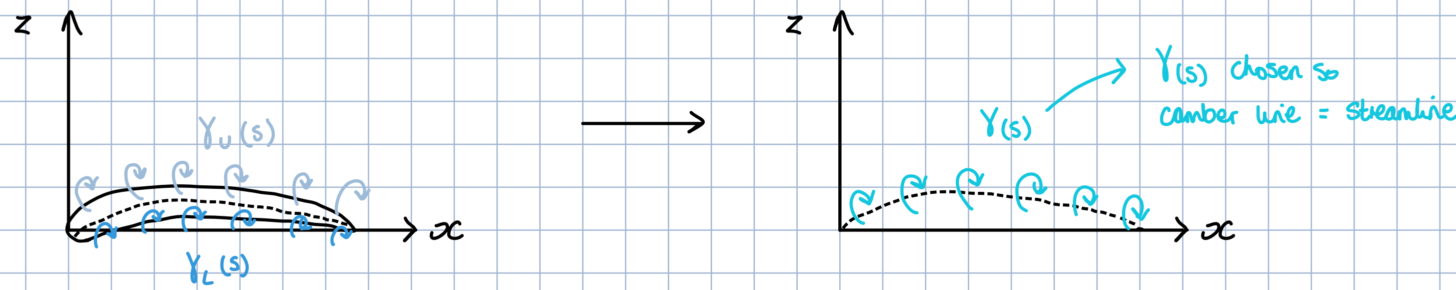
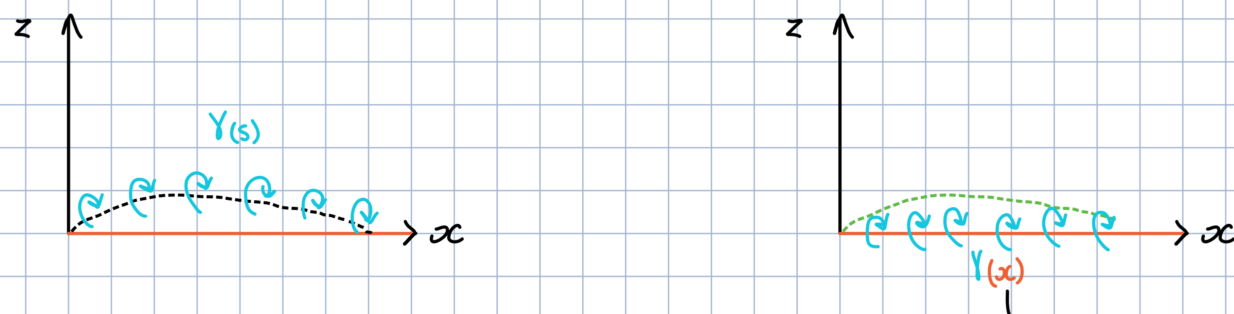
Thin Aerofoil Assumptions :

1. Upper & lower surfaces become indistinguishable (far away)
→ use a single vortex sheet situated on mean camber line



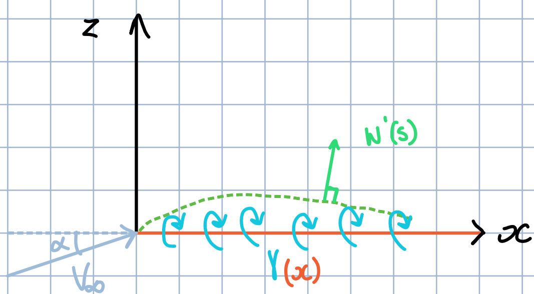
2. If camber small, vortex sheet can be moved to chordline



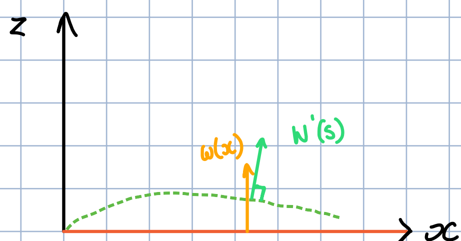
still choose $\gamma(x)$ so that mean camber line is streamline

- To make mean camber line a streamline, we need to work out :

→ comp. of freestream velocity normal to camber line
→ vortex-sheet induced component normal to camber line, $w'(s)$



3. As camber is small, Vortex-induced velocity on camber line ≈ Vortex-induced velocity on chord line



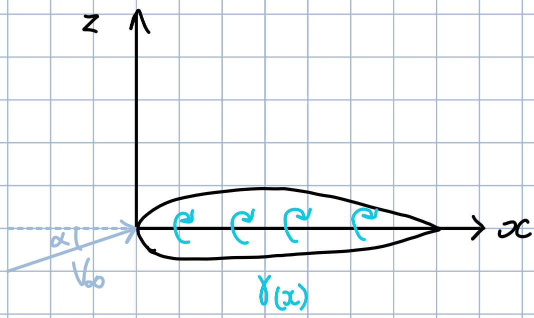
$$w'(s) \approx w(x)$$

- Kutta condition implies upper & lower surface velocities are equal at the trailing edge :

$$\boxed{\gamma(TE) = 0}$$

Symmetric or Uncambered Aerofoils :

For symmetric / uncambered aerofoils, chord = camber line $\therefore$ assumptions 2 & 3 unecessary.

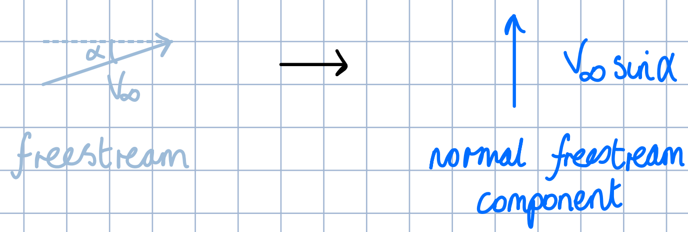


for symmetric aerofoil

$$\frac{dz}{dx} = 0$$

→ enforcing no normal flow, we must consider :

1. normal freestream component

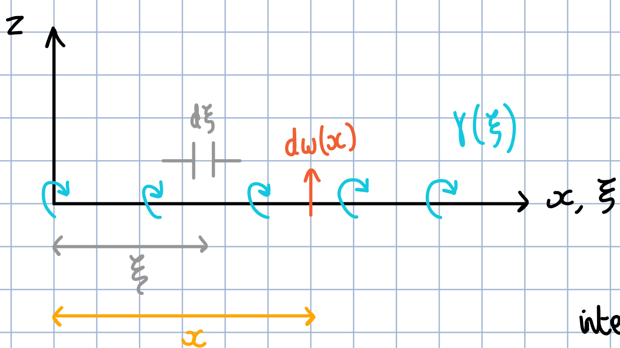


2. Vortex-induced normal

↳ due to symmetry $\quad w'(s) = w(x)$

- at a point x on chord line the vortex induced velocity is $w(x)$

→ found by considering velocity induced by a small element of the vortex sheet with strength $\gamma(\xi)\, d\xi$ located at a distance ξ from origin on chord line



$$V_\theta = -\frac{\Gamma}{2\pi r}$$

$$dw(x) = -\frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)}$$

integrate

$$\boxed{w(x) = -\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\, d\xi}{(x-\xi)}}$$

→ no normal flow condition is : $\quad V_\infty \sin\alpha + w(x) = 0$

for small angles : $\quad V_\infty \alpha + w(x) = 0$

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{(x-\xi)} = V_\infty \alpha$$

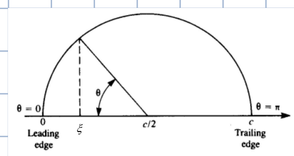
Fundamental thin aerofoil equation for symmetric case

↳ must be solved subject to Kutta condition

- To solve this equation, use transformation to change $\xi \rightarrow \theta$

$$\xi = \frac{c}{2}(1-\cos\theta)$$

$$x = \frac{c}{2}(1-\cos\theta_0)$$



→ substituting :

$$\boxed{\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\,d\theta}{\cos\theta-\cos\theta_0} = V_\infty\alpha}$$

don't need to be able to solve

θ is not polar angle, it's used to simplify integral

→ for a symmetric aerofoil

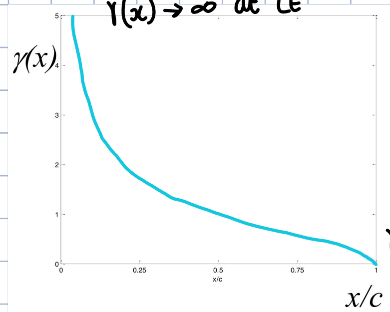
$$\boxed{\gamma(\theta) = 2\alpha V_\infty \frac{(1+\cos\theta)}{\sin\theta}}$$

→ to prove that it's the solution, we need to show it satisfies the equation :

1. It satisfies fundamental thin aerofoil equation
2. $\gamma(\pi) = 0$ → Kutta condition is satisfied

for transformation used TE at $\theta = \pi$

$\gamma(x) \rightarrow \infty$ at LE



$\gamma(x) \rightarrow 0$ at TE

Lift Distribution :

Method 1 :

Total Circulation $\boxed{\Gamma = \int_0^c \gamma(\xi)\,d\xi = \frac{c}{2}\int_0^\pi \gamma(\theta)\sin\theta\,d\theta = \pi\alpha c V_\infty}$

Do not write $\int_0^c \gamma(\theta)\,d\theta$ → disagrees with definition of γ as strength per unit length.

Find total lift using Kutta-Joukowski theorem

$$\boxed{L = \rho_\infty V_\infty \Gamma = \pi \alpha c \rho_\infty V_\infty^2}$$

$$C_L = \frac{L}{\frac{1}{2}\rho_\infty V_\infty^2 c} \longrightarrow \boxed{C_L = a_0 \alpha = 2\pi\alpha}$$

α in rads

$\boxed{a_0 = 2\pi}$ → gradient of C_L vs α graph for symmetric aerofoil

Method 2 :

- Consider infinitesimal element of sheet

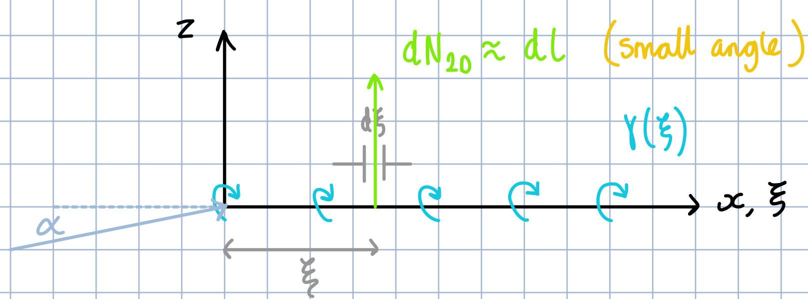
$$dL = \rho_\infty V_\infty d\Gamma = \rho_\infty V_\infty \gamma(\xi) d\xi$$

→ upon integration : $L = \rho_\infty V_\infty \Gamma = \pi \alpha c \rho_\infty V_\infty^2$

$$C_L = a_0 \alpha = 2\pi\alpha$$

Pitching Moment :

- Taking moments about leading edge :



lift defined normal to freestream but due to small angle α we can approx $dN \approx dL$

$$dM_{LE} = -\xi dL = -\xi \rho V_\infty d\Gamma = -\rho V_\infty \xi \gamma(\xi) d\xi$$

moments taken anticlockwise & +ve clockwise for aero

$$M_{LE} = -\rho V_\infty \int_0^c \xi \gamma(\xi) d\xi$$

$$\boxed{C_{M_{LE}} = \frac{M_{LE}}{\frac{1}{2}\rho V_\infty^2 c^2} = -\frac{\pi}{2}\alpha = -\frac{C_L}{4}}$$

two reference points :

$$\boxed{\bar{x}_{cp} = -\frac{C_{M_{LE}}}{C_L} = 0.25}$$

Centre of pressure : point about which moments are zero

→ Symmetric aerofoil

$$\boxed{\bar{x}_{ac} = -\frac{dC_{M_{LE}}}{dC_L} = 0.25}$$

aerodynamic centre : point about which moment is independent of angle of attack

Cambered Thin Aerofoils :

- For cambered aerofoil , $\boxed{dz/dx \neq 0}$

Fundamental thin aerofoil equation :

$$\boxed{V_\infty\left(\alpha + \frac{dz}{dx}\right) + w(x) = 0}$$

do not need to know how to derive or solve

Solution :

$$\boxed{\gamma(\theta) = 2V_\infty\left(A_0 \frac{(1+\cos\theta)}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin\theta\right)}$$

definition of θ same as symmetric case

→ $A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \frac{dz}{dx}\,d\theta_0$, $A_n = \frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos n\theta_0\,d\theta_0$

→ symmetric aerofoil special case where $A_0 = \alpha$ & A_n terms $= 0$

Circulation can shown to be :

$$\Gamma = \frac{c}{2}\int_0^\pi \gamma(\theta)\sin\theta\,d\theta = cV_\infty\left(\pi A_0 + \frac{\pi}{2}A_1\right)$$

only first two terms of solution affect lift

→ applying Kutta - Joukowski :

$$\boxed{C_L = \frac{\rho_\infty V_\infty \Gamma}{\frac{1}{2}\rho_\infty V_\infty^2 c} = \pi(2A_0 + A_1)}$$

→ substituting for the Fourier coefficients the lift coefficient is :

$$\boxed{C_L = a_0 (\alpha - \alpha_{L=0})}$$

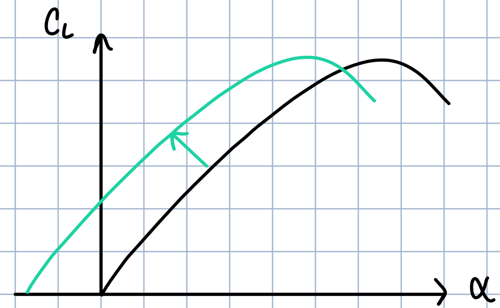
lift curve slope $a_0 = 2\pi$

$\alpha_{L=0}$ lift term due to camber → offset

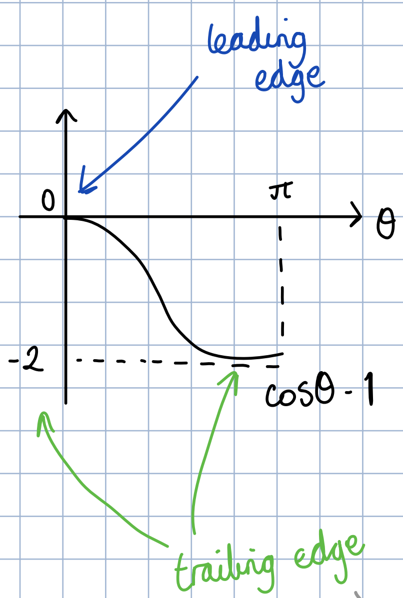
$$\boxed{\alpha_{L=0} = -\frac{1}{\pi}\int_0^{\pi} \left(\frac{dz}{dx}\right)(\cos\theta_0 - 1)\, d\theta_0}$$

Effect of Camber on Lift :

- +ve camber gives -ve offset → lift curve shifted upwards



- $\alpha_{L=0}$ depends on $(\cos\theta - 1)$



integrating $\frac{dz}{dx}$ at value of ~ -2 has much larger influence than at LE where value ~ 0 for $\alpha_{L=0}$ equation

↳ ∴ use of TE flaps used to increase lift at a given incidence by affecting $\alpha_{L=0}$

Effect of Camber on Pitching Moment :

- Similar analysis as before gives

$$C_{m_{LE}} = -\frac{\pi}{2}\left(A_0 + A_1 - \frac{A_2}{2}\right)$$

or in terms of lift coefficient

$$\boxed{C_{m_{LE}} = -\frac{C_L}{4} - \frac{\pi}{4}(A_1 - A_2)}$$

offset due to 'zero lift pitching moment' C_{m0}

- two moment reference points :

$$\bar{x}_{cp} = -\frac{C_{MLE}}{C_L} = 0.25\left\{1 + \frac{\pi}{C_L}(A_1 - A_2)\right\}$$

shifts with α

$$\bar{x}_{ac} = -\frac{dC_{MLE}}{dC_L} = 0.25$$ → at quarter chord

Lumped Vortex Method :

- Thin aerofoil theory method problem has continuous $\gamma(s)$ distribution that requires analytic solution

→ can this method be replaced with point or 'lumped' vortices, so that viewed from afar it predicts same lift as full thin aerofoil theory ?

- We develop general method like generalised 2D methods but with the location of singularities and control points linked to panel representation of the geometry.

→ camber line of thin aerofoil split into series of N flat panels with a point vortex on each

→ no normal flow condition applied on each panel and a set of equations solved for the vortex strengths in similar way to previous panel method solutions.

How do we get correct lift ?

→ Kutta Condition needs to be enforced :

we use Katz & Plotkin method

- When applying new method we're developing a thin aerofoil represented by N panels on camber line we will usually just

  - place the vortices
  - apply boundary condition
  - solve for vortex strengths

- Kutta condition not directly applied. Have already applied at TE in analytic thin aerofoil to get good lift predictions.

- Instead, match that lift from analytic by enforcing suitable choices for location of singularities & control points

  ↳ apply Kutta condition indirectly or implicitly

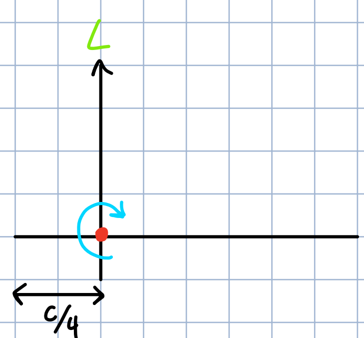
- Appropriate locations for singularities & control points established by looking at thin symmetric aerofoil with $N = 1$

→ we model so that this has same lift as thin aerofoil theory, since experiment agreement good.

→ this means that the circulation of the point vortex must equal the total circulation of thin aerofoil sheet.

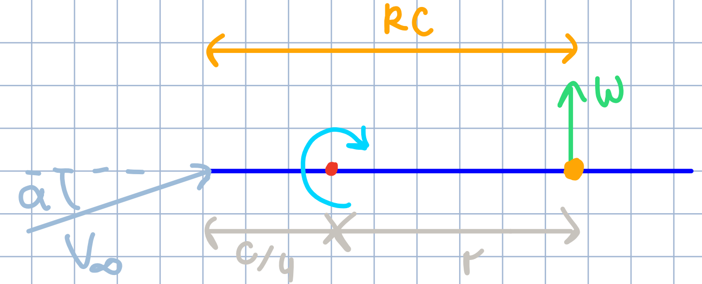
  ↳ we know vortex strength we want but need to know where the vortex and control point have been located to get this value as a solution of the problem:

- Since the lift of the thin symmetric aerofoil acts at the centre of pressure $\left(\frac{1}{4}c\right)$ the point vortex, strength Γ, is placed at the panel quater chord point.



- 'No normal flow' not applied at TE as lift prediction would be wrong

- We want to find a location of control point that gives us the known lift value:



No normal flow:

$$V_\infty \sin\alpha + w = 0 \quad \approx \quad V_\infty \alpha + w = 0$$

$$\boxed{w = V_\theta = \frac{-\Gamma}{2\pi r} = \frac{-\Gamma}{2\pi\left(kc - \frac{c}{4}\right)}}$$

→ this equation gives 2 unknowns Γ & k

but Γ same as thin aerofoil theory to give same lift:

for symmetric thin: $L = \rho_\infty V_\infty \Gamma = \pi \alpha c \rho_\infty V_\infty^2$

$$\therefore \boxed{\Gamma = \pi \alpha c V_\infty}$$

→ $\dfrac{-\pi c V_\infty \alpha}{2c\left(kc - c/4\right)} + V_\infty \alpha = 0 \quad \therefore \boxed{k = 0.75}$

3/4 chord point

- Control point often called 'aft neutral point'

→ effectively accounts for Kutta condition & gives surprisingly good results.

used for preliminary calcs.

detail of flow close to aerofoil is poor

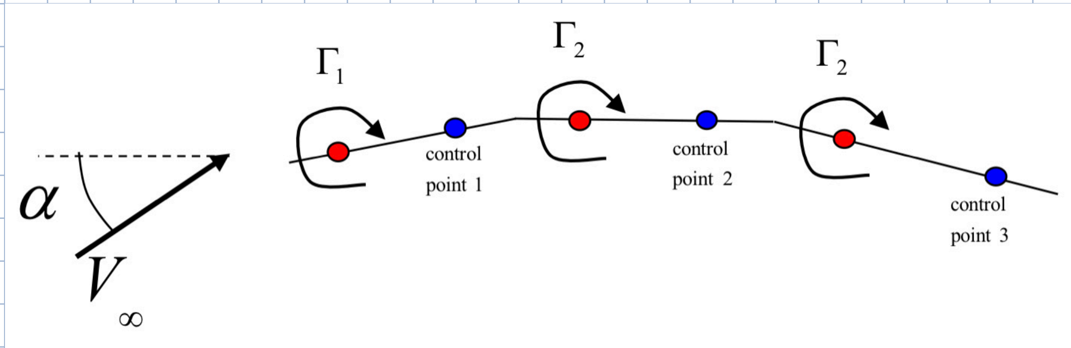
only valid in 2D

When extending to approximate flow past 'thin aerofoils' using a set of panels:

- each panel has:

→ point vortex at its quater chord

→ control point at three-quater chord



- More accurate than single point vortex but remains cheap, simple method.

Solved by calculation of influence coefficients & influence matrix equation.

→ solve linear set of equations then make secondary calculations.

$$A\vec{\Gamma} = \vec{R}$$

$\vec{\Gamma} = [\Gamma_1, \Gamma_2, \Gamma_3 \dots \Gamma_N]^T$

$\vec{R} = [R_1, R_2, R_3 \dots R_N]^T$ from freestream $R_i = -\vec{V}_\infty \cdot \vec{n}_i$

A is velocity influence matrix with entries $a_{i,j}$

→ for two panel examples, can solve using set of simultaneous equations rather than matrix.

Use of 'lumped vortex' method:

- Useful for multi-element configurations:
  - tandem wings
  - ground effect
  - wind tunnel interference

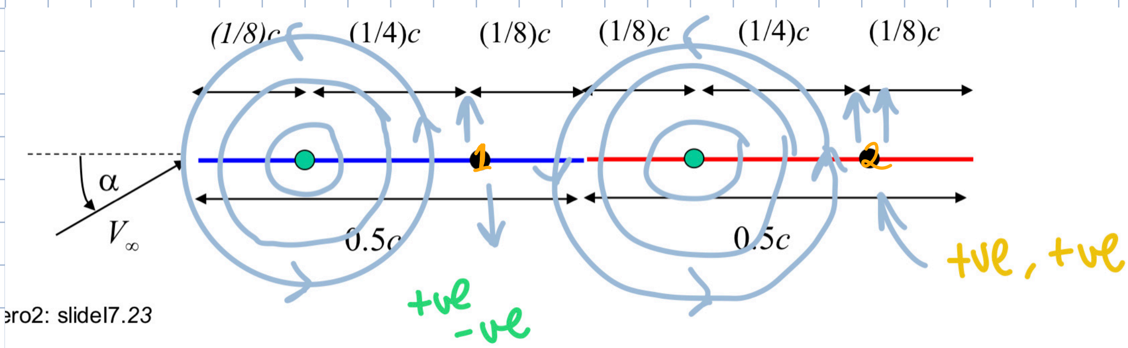
- When using method for such cases there are multiple vortices due to several aerofoils / wings being represented or there are image vortices.

reflected vortices in a plane

- the flow velocities induced by other velocities must be taken into account :
  - at control points → governs strength of vortices
  - at vortex locations → governs lift generated by vortices

Use of correct signs imperative for calculations :

e.g



at control point 1 : $V_\infty \sin\alpha + (V_\theta)_1 - (V_\theta)_2 = 0$

at control point 2 : $V_\infty \sin\alpha + (V_\theta)_1 + (V_\theta)_2 = 0$

assuming small angles :

① $$V_\infty \alpha + \left(\frac{-\Gamma_1}{2\pi\left(\frac{1}{4}\right)c}\right) - \left(\frac{-\Gamma_2}{2\pi\left(\frac{1}{4}\right)c}\right) = 0$$

② $$V_\infty \alpha + \left(\frac{-\Gamma_1}{2\pi\left(\frac{3}{4}\right)c}\right) - \left(\frac{-\Gamma_2}{2\pi\left(\frac{1}{4}\right)c}\right) = 0$$

$$\therefore \quad \Gamma_1 = \left(\frac{1}{4}\right)\pi V_\infty \alpha c$$

$$\Gamma_2 = \left(\frac{3}{4}\right)\pi V_\infty \alpha c$$

Lift given by $\quad l = \rho_\infty V_\infty (\Gamma_1 + \Gamma_2)$

$$\rightarrow \quad l = \rho_\infty V_\infty^2 \pi \alpha c \left(\frac{1}{4} + \frac{3}{4}\right) = \rho_\infty V_\infty^2 \pi \alpha c$$

∴ C_l same as thin aerofoil theory :

$C_l = 2\pi\alpha$